

A Parity Game Tale of Two Counters

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Abstract Parity games have important practical applications in formal verification and synthesis, especially for problems related to linear temporal logic and to the modal μ -calculus. The problem is believed to admit a solution in polynomial time, motivating researchers to find candidates for such an algorithm and to defeat these algorithms.

We present a parameterized parity game called the Two Counters game, which provides an exponential lower bound for a wide range of parity game solving algorithms. We are the first to provide an exponential lower bound to priority promotion with the delayed promotion policy, and the first to provide such a lower bound to tangle learning.

1 Introduction

Parity games are turn-based games played on a finite graph. Two players *Odd* and *Even* play an infinite game by moving a token along the edges of the graph. Each vertex is labeled with a natural number *priority* and the winner of the game is determined by the parity of the highest priority that is encountered infinitely often. Player Odd wins if this parity is odd; otherwise, player Even wins.

Parity games play a central role in several popular domains in theoretical computer science. Their study has been motivated by their relation to many problems in formal verification and synthesis that can be reduced to the problem of solving parity games, as parity games capture the expressive power of nested least and greatest fixpoint operators [13]. Deciding the winner of a parity game is polynomial-time equivalent to checking non-emptiness of non-deterministic parity tree automata [31], and to the explicit model-checking problem of the modal μ -calculus [11,26,30]. Synthesis problems ask whether an implementation of a system exists that satisfies the desired properties. Solving the related parity game either results in such an implementation or in a counterexample demonstrating how an adversary can make the property fail.

Parity games are also interesting in complexity theory, as the problem of determining the winner of a parity game is known to lie in $UP \cap co-UP$ [27], which is contained in $NP \cap co-NP$ [11]. This problem is therefore unlikely to be NP-complete and it is widely believed that a polynomial solution exists.

Recent work proposes novel parity game solving algorithms based on the notion of a tangle [8]. A tangle is a strongly connected subgraph of a parity game for

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which one of the players has a strategy to win all cycles in the subgraph. Tangles play a fundamental role in various parity game algorithms, but most algorithms are not explicitly aware of tangles and can explore the same tangles repeatedly, especially in the presence of *nested* tangles [8]. The algorithms proposed in [8] solve parity games by explicitly computing tangles using attractor computation.

Tangles are similar to snares [12] and quasi-dominions [4], with the critical difference that tangles are strongly connected whereas snares and quasi-dominions may be composed of multiple tangles. Thus it is an obvious question whether the superpolynomial counterexample of Friedmann [22] to Fearnley’s snare-based algorithm [12] can be adapted for tangle learning. We show that this is possible.

We propose a parameterized parity game based on binary counters. The goal is to trick the algorithms to explore the progression of the counters, only solving the game when all bits are set. The critical ingredient to make these games difficult for attractor-based algorithms is to use two intertwined binary counters, one for each player, that progress together. We show empirically that these games are difficult for a wide range of algorithms, in particular for those based on attractor computation, such as priority promotion [4,6] and its variations [2,3], tangle learning [8], the recursive algorithm by Zielonka [32,34], and the APT algorithm [33]. For these, we provide the exponential lower bound of $\Omega(2^{\sqrt{n}})$.

Notice that we are the first to provide an exponential lower bound for the delayed promotion policy of priority promotion and for tangle learning.

2 Preliminaries

Parity games are two-player turn-based infinite-duration games over a finite directed graph $G = (V, E)$, where every vertex belongs to exactly one of two players called player *Even* and player *Odd*, and where every vertex is assigned a natural number called the *priority*. Starting from some initial vertex, a play of both players is an infinite path in G where the owner of each vertex determines the next move. The winner of such an infinite play is determined by the parity of the highest priority that occurs infinitely often along the play.

More formally, a parity game \mathcal{G} is a tuple $(V_{\diamond}, V_{\square}, E, \text{pr})$ where $V = V_{\diamond} \cup V_{\square}$ is a set of vertices partitioned into the sets V_{\diamond} controlled by player *Even* and V_{\square} controlled by player *Odd*, and $E \subseteq V \times V$ is a left-total binary relation describing all moves, i.e., every vertex has at least one successor. We also write $E(u)$ for all successors of u and $u \rightarrow v$ for $v \in E(u)$. The function $\text{pr}: V \rightarrow \{0, 1, \dots, d\}$ assigns to each vertex a *priority*, where d is the highest priority in the game.

We write $\text{pr}(v)$ for the priority of a vertex v and $\text{pr}(V)$ for the highest priority of vertices V and $\text{pr}(\mathcal{G})$ for the highest priority in the game \mathcal{G} . Furthermore, we write $\text{pr}^{-1}(i)$ for all vertices with the priority i . A *path* $\pi = v_0 v_1 \dots$ is a sequence of vertices consistent with E , i.e., $v_i \rightarrow v_{i+1}$ for all successive vertices. A *play* is an infinite path. We denote with $\text{inf}(\pi)$ the vertices in π that occur infinitely many times in π . Player Even wins a play π if $\text{pr}(\text{inf}(\pi))$ is even; player Odd wins if $\text{pr}(\text{inf}(\pi))$ is odd. We write $\text{Plays}(v)$ to denote all plays starting at vertex v .

A *strategy* $\sigma: V \rightarrow V$ is a partial function that assigns to each vertex in its domain a single successor in E , i.e., $\sigma \subseteq E$. We refer to a strategy of player α to restrict the domain of σ to V_α . In the remainder, all strategies σ are of a player α . We write $\text{Plays}(v, \sigma)$ for the set of plays from v consistent with σ , and $\text{Plays}(V, \sigma)$ for $\{\pi \in \text{Plays}(v, \sigma) \mid v \in V\}$.

A fundamental result for parity games is that they are memoryless determined [10], i.e., each vertex is either winning for player Even or for player Odd, and both players have a strategy for their winning vertices. Player α wins vertex v if they have a strategy σ such that all plays in $\text{Plays}(v, \sigma)$ are winning for player α .

A *dominion* D is a set of vertices for which player α has a strategy σ such that all plays consistent with σ stay in D and are winning for player α . That is, all cycles in the induced subgame of σ are won by player α . We also write a *p-dominion* for a dominion where p is the highest priority encountered infinitely often in plays consistent with σ , i.e., $p := \max\{\text{pr}(\inf(\pi)) \mid \pi \in \text{Plays}(D, \sigma)\}$.

Several algorithms for solving parity games employ *attractor computation*. Given a set of vertices A , the attractor of A for a player α represents those vertices from which player α can ensure arrival in A . We write $\text{Attr}_\alpha^\ominus(A)$ to attract vertices in \ominus to A as player α , i.e.,

$$\mu Z . A \cup \{v \in V_\alpha \mid E(v) \cap Z \neq \emptyset\} \cup \{v \in V_{\bar{\alpha}} \mid E(v) \subseteq Z\}$$

Informally, we compute the α -attractor of A with a backward search from A , initially setting $Z := A$ and iteratively adding α -vertices with a successor in Z and $\bar{\alpha}$ -vertices with no successors outside Z .

Attractors are often used to attract to a set $A := \text{pr}^{-1}(\text{pr}(\ominus))$ for the player that wins priority $\text{pr}(\ominus)$, i.e., the highest priority vertices in \ominus . By repeatedly computing this attractor and removing it from the game, the game is decomposed into *regions*. The recursive algorithm investigates whether lower regions attract vertices from higher regions of the opponent. Priority promotion merges suitable lower regions with higher regions. Tangle learning analyses suitable regions to compute tangles which improve the decomposition.

3 Tangle Learning

Earlier work introduced tangles as substructures of parity games [8]. Tangles are strongly connected subgraphs of a parity game for which one player has a strategy to win all cycles in the subgraph. The losing player must therefore escape the tangle, so we extend attractor computation to simultaneously attract all vertices in a tangle when the losing player must escape to the attracting set. This leads to the *tangle learning* algorithm, which computes new tangles along the top-down decomposition of the game, computed using the extended attractor. The algorithm solves parity games by finding tangles that are dominions.

Definition 1. A *p-tangle* is a nonempty set of vertices $U \subseteq V$ with $p = \text{pr}(U)$, for which player $\alpha \equiv_2 p$ has a strategy $\sigma: U_\alpha \rightarrow U$, such that the graph (U, E') , with $E' := E \cap (\sigma \cup (U_{\bar{\alpha}} \times U))$, is strongly connected and player α wins all cycles in (U, E') .

We have several basic observations related to tangles [8].

1. A p -tangle from which player $\bar{\alpha}$ cannot leave is a p -dominion.
2. Every p -dominion contains one or more p -tangles.
3. Tangles may contain tangles of a lower priority.

We can find a hierarchy of tangles in any dominion D with winning strategy σ by computing the set of winning priorities $\{\text{pr}(\inf(\pi)) \mid \pi \in \text{Plays}(D, \sigma)\}$. There is a p -tangle in D for every p in this set. Tangles are thus a natural substructure of dominions. One can find all tangles in a dominion D by computing $\{\inf(\pi) \mid \pi \in \text{Plays}(D, \sigma)\}$.

See for example Figure 1. Player Odd wins with highest priority 5 and strategy $\{\mathbf{d} \rightarrow \mathbf{e}\}$. Player Even can also avoid priority 5 and then loses with priority 3. The 5-dominion $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$ contains the 5-tangle $\{\mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}\}$ and the 3-tangle $\{\mathbf{c}, \mathbf{e}\}$.

As described in [8], tangles play a central role for various parity game solving algorithms, as they implicitly explore tangles and may explore the same tangles repeatedly, especially when tangles are nested. This motivates algorithms that explicitly target tangles, such as the “snare memoization” extension to strategy improvement [12] and the “tangle attractor” approach of tangle learning [8].

Tangle learning is based on the tangle attractor. We write $E_T(t)$ for the edges from $\bar{\alpha}$ -vertices in the tangle t to the rest of the game: $E_T(t) := \{v \mid u \rightarrow v \wedge u \in t \cap V_{\bar{\alpha}} \wedge v \in V \setminus t\}$. We extend attractor computation to attract vertices of tangles won by player α where the losing player must play to the attracting set, writing $TAttr_{\alpha}^{\bar{\alpha}, T}(A)$ to attract vertices in \mathcal{D} and vertices of tangles in the set T that are in \mathcal{D} to A as player α , i.e.,

$$\begin{aligned} \mu Z . A \cup \{ v \in V_{\alpha} \mid E(v) \cap Z \neq \emptyset \} \cup \{ v \in V_{\bar{\alpha}} \mid E(v) \subseteq Z \} \\ \cup \{ v \in t \mid t \in T \wedge t \subseteq V \wedge \text{pr}(t) \equiv_2 \alpha \wedge E_T(t) \subseteq Z \} \end{aligned}$$

The tangle learning algorithm repeatedly decomposes the game with the tangle attractor. By computing the bottom strongly connected components of the regions, new tangles are obtained and added to the set of tangles. Each iteration of this algorithm adds new tangles to this set, resulting in a different decomposition each time. Tangles without escapes are dominions, which are then maximized using the attractor and removed from the game.

4 Priority Promotion

Priority promotion was proposed in [4,6] and improved in [2,3]. Priority promotion computes a top-down α -maximal decomposition of the game into regions associated with a so-called *measure*, which is the highest priority in the region.

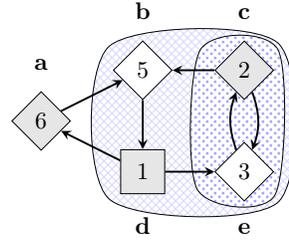


Figure 1. A 5-dominion with a 5-tangle and a 3-tangle

These regions have the property that all plays that stay in the region are won by player α . Regions are *closed* when all vertices of player α have a successor in the region and no vertices of player $\bar{\alpha}$ have successors in lower regions. Closed regions can be merged with the lowest higher region to which player $\bar{\alpha}$ can escape, after which the decomposition of the game is refined by attracting to this merged region and recomputing the decomposition of the lower regions. This is called promoting, as the measure of the lower region is “promoted” to the measure of the higher region. Notice that a closed α -region is essentially a collection of (possibly unconnected) tangles and vertices that are attracted to these tangles.

The original priority promotion algorithm [4] resets all lower regions after promotion by recomputing the decomposition. The decomposition of lower regions is affected when the merged region attracts vertices from lower regions that were not attracted before. The problem here is that the lower regions might be the result of earlier promotions and these steps are often repeated after recomputing the decomposition. The PP+ extension [2] only resets lower regions of player $\bar{\alpha}$, since promoting regions of player α can only potentially “break” regions of player $\bar{\alpha}$. The DP extension (also [2]) uses a heuristic to delay certain promotions. If a promotion might affect a merged region, then this promotion is delayed. The delayed promotions are instead performed on a copy of the decomposition. When no more normal promotions can be performed, the delayed promotions of one player are applied from this copy, where this player is the one that wins the highest merged region in the copy. The region restoration (RR) extension [3] further improves on PP+ by only resetting lower regions of either player if the original attractor strategy of the player for the vertices that are still in the original region now leaves this region to a higher region (of the opponent).

5 A Tale of Two Counters

The design of our parameterized parity game is loosely inspired upon earlier work by Friedmann [22], which provides a superpolynomial lower bound for the non-oblivious strategy improvement algorithm [12]. This algorithm is a variation of strategy improvement that learns snares, which are related to tangles. However, attractor-based algorithms require a different approach, because attractor-based algorithms trivially solve these counterexamples.

Each bit k of our counter is a substructure of $4 + 3k$ many vertices that contains 2^k many tangles. A bit is set by learning one of these tangles and attracting an “input” vertex. We use two counters, one for each player, that are intertwined and progress together. The bits are connected to the input vertices of other bits such that this tangle is no longer attracted when the higher bits change, thus resetting the bit. Furthermore, if lower bits are not set, then the tangle cannot be found because the player is distracted by a higher priority.

See Fig. 2 for bit 2 of some player α in a 5-bit counter. The highest bit is bit 0 and the lowest bit is bit 4. Vertex **i** is the “input” vertex and has a high priority of player $\bar{\alpha}$ ’s parity. Vertex **h** is the “high” vertex and has a high priority of player α ’s parity. Vertex **t** is the “tangle” vertex and has a low priority of player α ’s

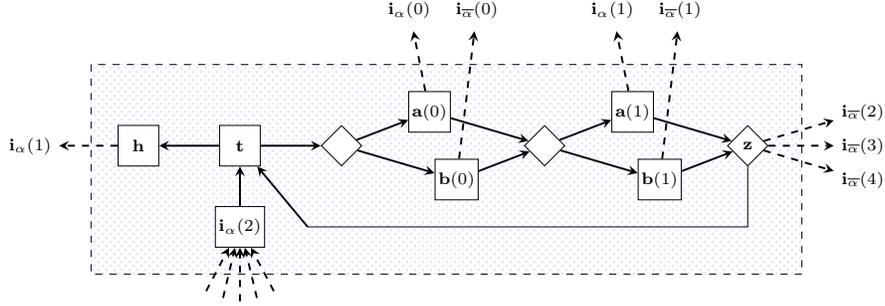


Figure 2. Bit 2 of a 5-bit Two Counters game. We use diamonds for vertices of player α and boxes for vertices of player $\bar{\alpha}$.

Player	Bit	t	i	h	Edge $z(b) \rightarrow i_{\bar{\alpha}}(b)$?
Even	bit 0	2	13	20	yes
Odd	bit 0	3	12	19	no
Even	bit 1	4	11	18	yes
Odd	bit 1	5	10	17	no
Even	bit 2	6	9	16	yes
Odd	bit 2	7	8	15	no

Table 1. Instantiation of a Two Counters game with $N = 3$, where player Odd starts.

parity. Vertices \mathbf{h} and \mathbf{i} of higher bits have higher priorities, while vertices \mathbf{t} of higher bits have lower priorities. All other vertices have priority 0.

Vertex \mathbf{t} is the highest priority vertex of the tangles. When the bit is set by learning a tangle, this tangle is attracted to vertex \mathbf{h} and attracts input \mathbf{i} to \mathbf{h} . Thus, vertex \mathbf{i} is in a region of player α if the bit is set, and in a region of player $\bar{\alpha}$ if the bit is not set. That is, vertex \mathbf{i} is good for player α if the bit is set, and for player $\bar{\alpha}$ otherwise. Vertex \mathbf{h} has an edge to the input of the next higher bit, except vertex \mathbf{h} of the highest bit has an edge to the input of the lowest bit.

To find a tangle, player α must force their opponent to play from \mathbf{t} to \mathbf{z} such that they can only escape to \mathbf{h} or to higher regions of player α . Each tangle in the bit uses a different path from \mathbf{t} to \mathbf{z} , depending on the state of the higher bits. If the two counters progress together, then either $\mathbf{i}_{\alpha}(0)$ is good for α and $\mathbf{i}_{\bar{\alpha}}(0)$ is bad for α , or vice versa. In Fig. 2, if bit 0 is set and bit 1 is unset, then the path via $\mathbf{a}(0)$ and $\mathbf{b}(1)$ forces the opponent to stay in the tangle, whereas they could play to their own region via $\mathbf{b}(0)$ and $\mathbf{a}(1)$. Furthermore, player α must choose to play from \mathbf{z} to \mathbf{t} . When lower $\bar{\alpha}$ -bits are not set, the $\mathbf{i}_{\bar{\alpha}}$ vertices are in a region of player α higher than \mathbf{t} and are thus more attractive to play to. Finally, we introduce an asymmetry, since one player starts counting first having the lowest priority \mathbf{i} and \mathbf{h} vertices. We only have the edge from vertex \mathbf{z} to the matching vertex $\mathbf{i}_{\bar{\alpha}}$ for the bits of the second player. This forces the second player to wait until the first player sets the corresponding bit.

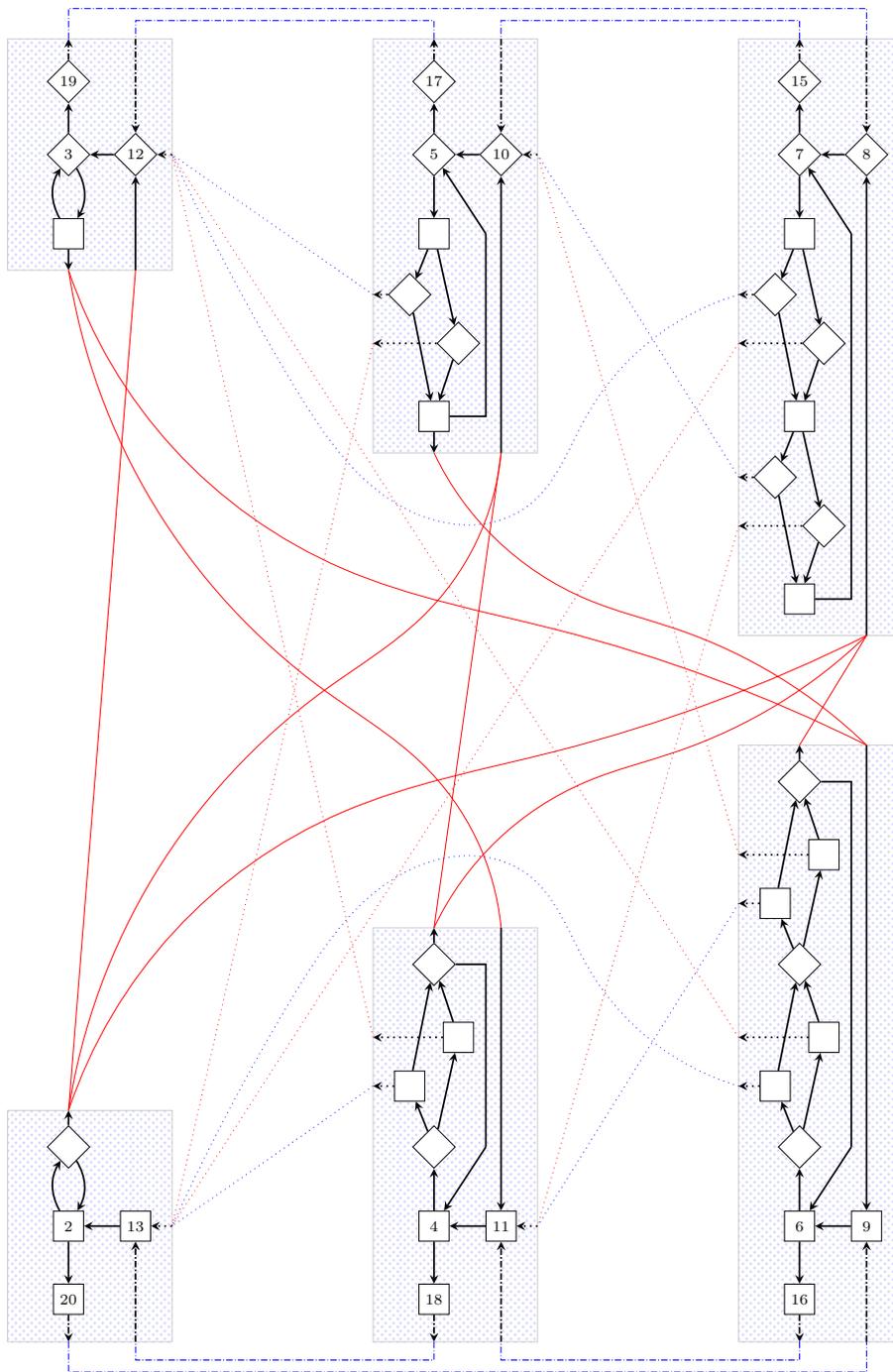


Figure 3. The 3-bit Two Counters game. The unlabeled vertices have priority 0.

See Table 1 for an example with 3 bits, and Fig. 3 for the full 3-bit Two Counters game. Player Odd is the first to set bit 2, followed by player Even. When all bits are set, both players have a dominion consisting of their entire counter via the blue edges.

The number of vertices for a Two Counters game with N bits for both players is $3N^2 + 5N$ and the number of edges is $7N^2 + 4N$.

6 Empirical evaluation

To assess the runtime complexity of solving N -bit Two Counters games with different algorithms, we solve them and obtain a relevant statistic that is indicative of the runtime. We support the claim that Two Counters games require time exponential in the parameter N for these solvers based on this statistic. Although this is a weaker basis than a full complexity proof, a full complexity proof is more difficult to produce and to understand, while the exponential behavior is clear from the empirical data presented in this section and in Section 7.

To solve the games we use the implementation in the parity game solver Oink [9], but we have also confirmed the results with implementations in PG-Solver [24]. Apart from the earlier described algorithms, we consider the APT algorithm [33], the small progress measures algorithm [28] (SPM), and we also look at recent quasi-polynomial algorithms, namely the first idea by Calude et al. [7] as implemented as ordered progress measures by Fearnley et al. [14] (QPT) and the algorithm based on universal trees called succinct progress measures by Jurdziński et al. [29] (SSPM).

We report the following statistic for each solver:

- number of tangles for (alternating) tangle learning (TL, ATL)
- number of promotions for priority promotion (PP, RR, DP, RRDP)
- number of recursive calls for Zielonka’s recursive algorithm (ZLK)
- number of iterations (lower resets) for the APT algorithm
- number of lifts for progress measures algorithms (SPM, SSPM, QPT)

See Table 2. This table shows that the relevant statistic doubles with each higher number of bits, which supports that the runtime is exponential in the number of bits. The number of non-dominion tangles for ATL and TL is the same as the number of promotions for RR and RRDP, namely $2 \times (2^N - 1)$ tangles to set all bits. Since the number of vertices and edges increases quadratically, we conclude that the Two Counters games provide an exponential lower bound to these algorithms, namely $\Omega(2^{\sqrt{n}})$. We also investigated whether inflation and compression [9] had any effect and this was not the case. See further Section 7 for a closer look at how these algorithms solve the Two Counters games.

For the algorithms in Table 3, we use compression to obtain the lowest numbers. We do not investigate strategy iteration, as for every game there is a variation of the strategy iteration algorithm that will solve it in a linear number of steps [18, Lemma 4.2] and therefore designing a parity game that is difficult for *every* variation of strategy iteration is not possible. We also tried with the succinct progress measures where we restrict the size of the used universal tree,

bits	n	ATL tangles	TL	PP	DP promotions	RR	RRDP	ZLK calls
1	8	2	2	2	2	2	2	8
2	22	6	6	9	7	6	6	21
3	42	14	14	23	18	14	14	45
4	68	30	30	52	43	30	30	91
5	100	62	62	112	97	62	62	181
6	138	126	126	235	210	126	126	359
7	182	254	254	485	442	254	254	713
8	232	510	510	990	913	510	510	1,419
9	288	1,022	1,022	2,006	1,863	1,022	1,022	2,829
10	350	2,046	2,046	4,045	3,772	2,046	2,046	5,647
15	750	65,534	65,534	130,961	122,742	65,534	65,534	180,249
20	1,300	2,097,150	2,097,150	4,194,108	3,931,927	2,097,150	2,097,150	5,767,203

Table 2. Solving the Two Counters games with different algorithms. We report the relevant statistic that represents the runtime for each algorithm.

bits	n	APT iterations	SPM lifts	SSPM lifts	QPT lifts	SSPM k=2	QPT k=3
1	8	8	31	137	76	49	44
2	22	44	221	20,506	7,776	342	259
3	42	172	842	590,341	563,160	973	670
4	68	1,058	3,407	14,783,111	3,196,795	2,375	1,534
5	100	5,776	11,011	61,722,742	206,491,344	4,490	2,860
6	138	20,392	40,051		757,953,011	8,363	5,292
7	182	114,752	145,683			13,328	8,355
8	232	608,736	557,040			20,897	12,815
9	288	1,748,160	2,172,283			30,607	18,231
10	350	19,546,240	8,577,454			43,842	25,353

Table 3. Solving the Two Counters games with different algorithms. We report the relevant statistic that represents the runtime for each algorithm.

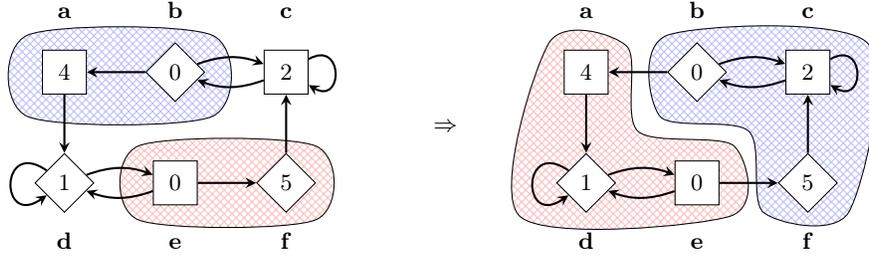


Figure 4. Vertex **a** is a distraction. The distraction is removed by the opponent’s tangle $\{d\}$. We can then learn tangle $\{b, c\}$. Similarly, vertex **f** distracts tangle $\{d, e\}$.

that is, we use 2-bounded adaptive counters, and with the ordered progress measures where we restrict the tuples to 3 components. We then successfully solve the games rather quickly. Without this modification, the algorithms did not finish within an hour for larger instances.

7 Analysis

We study how the three attractor-based algorithms and the APT algorithm solve a Two Counters game consisting of 3 bits.

7.1 Distractions

We introduce the notion of a distraction as a way to understand parity game solving algorithms. Assume that we are running an attractor-based algorithm and computed the attractor set Z to the highest vertices of a (sub)game \mathcal{D} .

A distraction is a vertex v with the highest priority of region Z , where the opponent’s winning region in $\mathcal{D} \setminus Z$ directly attracts v . This means that the opponent has a tangle in $\mathcal{D} \setminus Z$ that attracts v and this tangle is either a dominion or it is necessarily attracted to a higher region of the opponent. Initially player α believes they should play to v , but after solving the subgame or finding the attracting tangle in the subgame, playing to v is actually good for the opponent.

We say that a distraction v *distracts a tangle* t , with $v \notin t$, if there is some *distracted vertex* $w \in t$, that is attracted by v , but that can also avoid v to be part of tangle t , such that the tangle avoids v . The player α that wins the distracted tangle also controls the distracted vertex and v has a higher priority of α ’s parity. In order to learn the distracted tangle, the solver first needs to “remove” the distraction. As argued in [8], tangle learning removes distractions by learning the opponent’s tangle that attracts the distraction. See for example Fig. 4. Vertices **a** and **f** are distractions, as they lead to tangles won by the opponent.

Zielonka’s recursive algorithm and priority promotion also rely on this mechanism to remove distractions. This is most explicit in Zielonka’s recursive algorithm as described in [9], as the second recursive call is only done if the opponent

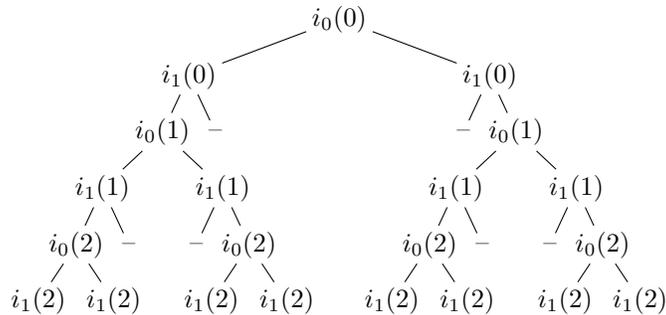
attracts from the current player's region, which is precisely when there is a distraction. Thus, distractions make the recursive algorithm slow, especially when the game is such that lower distractions must be removed in both recursive subgames of a higher distraction, i.e., in the first subgame to remove the higher distraction and in the second subgame after the distraction has been removed. If there are no distractions in the game, then a second recursive subgame is never solved and the recursive algorithm would run in at most n recursive calls.

7.2 Recursive algorithm

We refer to [9] for a full description of the recursive algorithm. As argued in Section 7.1, the recursive algorithm runs in exponential time if there are many distractions, and if they are both distracting before and after removing a higher distraction. The input vertices $\mathbf{i}_\alpha(b)$ are distractions for vertices \mathbf{z} of higher bits; whenever a solved subgame attracts a vertex \mathbf{i} to a region of the opponent, i.e., a bit is set, then many distractions of lower regions distract again. We can see this clearly from a trace of the recursive algorithm on the 3-bit TC game, which records the following timeline of removed distractions:

$$\begin{array}{ccccccc}
 i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow & \\
 i_1(1) & \Rightarrow & i_0(1) & \Rightarrow & i_1(1) & \Rightarrow & \\
 i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow & \\
 i_1(0) & \Rightarrow & i_0(0) & \Rightarrow & i_1(0) & \Rightarrow & \\
 i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow & \\
 i_1(1) & \Rightarrow & i_0(1) & \Rightarrow & i_1(1) & \Rightarrow & \\
 i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow &
 \end{array}$$

The pattern is clearer when displayed as a tree. Notice that this tree is similar to the recursive calls of the algorithm.



From the above tree it is clear that lower bits are set repeatedly by the algorithm and that the input vertices are repeatedly distracting as described.

Furthermore, as removing each distraction attracts a vertex from each lower bit and stops attracting a vertex from each opponent's lower bits, all subgames before and after removing the distraction are different. There are thus exponentially many distinct subgames to be solved.

7.3 Tangle learning

As explained in Section 5, the idea is that tangle learning follows the progression of the counters. When solving the 3-bit TC game with tangle learning, we see that the following tangles are learned in the 15 iterations:

1.	Odd bit 2	2.	Even bit 2
3.	Odd bit 1	4.	Even bit 1
5.	Odd bit 2	6.	Even bit 2
7.	Odd bit 0	8.	Even bit 0
8.	Odd bit 2	10.	Even bit 2
11.	Odd bit 1	12.	Even bit 1
13.	Odd bit 2	14.	Odd dominion, Even bit 2
15.	Even dominion		

Tangle learning thus solves the Two Counters games as expected.

7.4 Priority promotion

In priority promotion, a bit is set when the region of tangle vertex \mathbf{t} is closed, promotes to a higher region and attracts input vertex \mathbf{i} . This resets lower regions. When solving the 3-bit TC game with the delayed promotion policy, we see the following sequence of promotions:

	Promotion	Bit	Counter state (DP)		Delayed	Recovered
			odd	even		
1	7 to 11	Odd 2	1	0		
2	6 to 10	Even 2	1	1		
3	5 to 13	Odd 1	3	0	yes	
4	4 to 12	Even 1	2	2		
5	7 to 13	Odd 2	3	0	yes	
6	4 to 12	Even 1	3	2		yes
7	6 to 12	Even 2	3	3		
8	3 to 19	Odd 0	7	0	yes	
9	2 to 20	Even 0	0	4		
10	3 to 19	Odd 0	4	4		yes
11	7 to 11	Odd 2	5	4		
12	6 to 10	Even 2	5	5		
13	5 to 19	Odd 1	7	4	yes	
14	4 to 20	Even 1	0	6		
15	3 to 19	Odd 0	4	6		yes
16	5 to 19	Odd 1	6	6		yes
17	7 to 19	Odd 2	T	6		
18	6 to 20	Even 2	T	T		

We record the state of the two counters after each promotion with the DP solver. Notice that the counter for player Odd increases first. Setting odd bits never resets lower odd bits, but they are reset with the promotion of the next even

bit. We denote with \top that the dominion is found. The “recovered” promotions do not occur in the RR and RRDP algorithms, as their regions are recovered. All delayed promotions are immediately applied. Thus the delayed promotion policy is not useful here. We do see that region recovery is useful. Both PP and DP (based on PP+) perform additional promotions that are not necessary with region recovery. For the 3-bit TC game, PP requires 23 promotions, DP requires 18 promotions, RR requires 14 promotions and RRDP requires 14 promotions.

7.5 APT

See [33] for a description of the APT algorithm and the source code of Oink [9] for an efficient implementation in a single loop. The APT algorithm is described as a nested fixed point computation that refines “Avoiding” and “Visiting” sets. These two sets essentially encode the current knowledge of whether vertices with an odd priority are winning for odd and vertices with an even priority are winning for even, that is, whether vertices are distractions. Initially no vertex is considered a distraction. Each time some vertices of priority p are now considered a distraction, all vertices with a lower priority than p are reset. We look at the trace of how often the algorithm determines that a distraction in the 3-bit TC game must be avoided. This happens $31 \times$ in the following sequence:

$$\begin{array}{ccccccc}
 & i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow \\
 i_1(1) \Rightarrow i_1(2) & \Rightarrow & & i_0(1) \Rightarrow i_0(2) & \Rightarrow & i_1(1) & \Rightarrow \\
 & i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow \\
 i_1(0) \Rightarrow i_1(2) \Rightarrow i_1(1) \Rightarrow i_1(2) & \Rightarrow & i_0(0) \Rightarrow i_0(2) \Rightarrow i_0(1) \Rightarrow i_0(2) & \Rightarrow & i_1(0) & \Rightarrow \\
 & i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow \\
 i_1(1) \Rightarrow i_1(2) & \Rightarrow & & i_0(1) \Rightarrow i_0(2) & \Rightarrow & i_1(1) & \Rightarrow \\
 & i_1(2) & \Rightarrow & i_0(2) & \Rightarrow & i_1(2) & \Rightarrow
 \end{array}$$

This is clearly similar to the timeline of the recursive algorithm.

8 Discussion

Since solving parity games is known to lie in $UP \cap co-UP$, it is widely believed to admit a polynomial solution. Researchers have been studying this problem for several decades. For some time, the focus was on variations of strategy iteration algorithms, where a suitable *improvement rule* could lead to a polynomial solution. Friedmann et al. provided superpolynomial or exponential lower bounds for many of such rules [1,16,17,20,21,22,23], as well as Fearnley and Savani recently [15].

For a long time, the main attractor-based algorithm was the recursive algorithm by McNaughton [32] and Zielonka [34], for which Friedmann also showed an exponential lower bound [19] and later Gazda and Willemse [25] improved upon this lower bound. These lower bounds are resistant against techniques like inflation, compression and SCC decomposition, even when applied to each recursive call. Observing that the lower bound of Friedmann can be defeated using memoization, since the number of distinct solved subgames is polynomial,

Benerecetti et al. proposed a lower bound [5] that is resilient against memoization as well as the other techniques. Their priority promotion algorithm [4] solves these three lower bounds in polynomial time, although this requires inflation for Gazda’s lower bound. For the original algorithm, two exponential lower bounds are known [5,6]. For the variations of priority promotion, in particular for the delayed promotion policy [5], no lower bound has been published in the literature. Tangle learning [8] solves all these lower bound examples in polynomial time.

We presented the parameterized Two Counters game and showed that this provides an exponential lower bound for the recursive algorithm, for priority promotion, in particular with the delayed promotion policy, for tangle learning, and for the APT algorithm. An N -bit Two Counters game has size $n = 3N^2 + 5N$ and requires at least $2 \times (2^N - 1)$ steps, thus providing a lower bound of $\Omega(2^{\sqrt{n}})$. We are investigating variations of tangle learning that can effectively solve Two Counters games and whether we can defeat these variations is future work.

The Two Counters game is implemented in Oink [9] and is available online via <https://www.github.com/trolando/oink>.

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